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LETTER TO THE EDITOR

Singularities of normal forms and topology of orbits in area-preserving maps

A Bazzani† and G Turchetti‡

† Dipartimento di Matematica, Università di Bologna and INFN sezione di Bologna, Piazza di P San Donato 5, 40126 Bologna, Italy

‡ Dipartimento di Fisica, Università di Bologna and INFN sezione di Bologna, via Irnerio 46, 40126 Bologna, Italy

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Abstract. The existence of analytic normal forms for an area-preserving map with an elliptic fixed point is considered. If the linear frequency is diophantine, the complexified map can be analytically conjugated with an integrable map in a disc of the complex radial coordinate r excluding a sequence of sets, whose measure decreases exponentially fast as we approach the origin; the analyticity in the angle θ is just a strip. The non-analyticity domains correspond to the regions where the topology of the orbits changes since nonlinear resonances are present and the modulus of the residue of the leading poles, provided by perturbation theory, is the square of the width of the corresponding chains of islands. This suggests a relation between clusters of singularities of the normalizing transformation and changes in the topology of the orbits.

Integrability remains one the basic problems for Hamiltonian systems, since Poincaré [1] proved the non-existence of analytic first integral of motion beyond the Hamiltonian itself and Siegel [2] showed that this property is generic. The KAM theory [3–5] has shown that invariant manifolds like tori locally exist, for slightly perturbed integrable systems and that their measure is large for small perturbations.

On the other hand more recently Nekhoroshev [6] proved the asymptotic character of perturbative solution providing optimal estimates for the remainders and related stability results for exponentially long (in the inverse of perturbation) times.

The above picture is not at all contradictory; indeed what Poincaré and Siegel claim is the non-existence of analytic first integrals defined in open sets (such as polydiscs in cartesian coordinates). This is consistent with the divergence of Birkhoff series and with the local existence of real first integrals in Cantor-like sets, dictated by the KAM theory.

A unifying picture is obtained for Hamiltonian systems with two degrees of freedom by considering an area-preserving map with an elliptic fixed point with linear diophantine frequency and showing that analytic first integrals of motion can indeed be defined not in a polydisc, but in its complement with respect to a Cantor-like set intersecting the real phase space. The technique used to obtain this result is a combination of perturbation theory, Nekhoroshev-like remainder estimates and Newton's techniques borrowed from KAM theory for analyticity detection and analytic continuation.

The singularity structure is also investigated in more detail by using a functional equation for the Fourier components. This equation, proposed by Siegel and Moser

[7] in order to prove the convergence of the normal forms in the hyperbolic case, allows to establish the convergence for any complex fixed point and shows that the loss of analyticity in the elliptic case is due exclusively to the small divisors, namely to the nonlinear resonances.

By iterating this equation a sequence of non-perturbative approximations (with memomorphic functions) is obtained, whose singularities are poles corresponding to the nonlinear resonances. This result was first suggested by a leading divisors asymptotic analysis [8, 9]. To any nonlinear resonance there corresponds a pole in the real or imaginary r axis [10]; the real poles are located at the centre of the corresponding chains of islands and the residue of the poles is the square of the width of the islands themselves. This picture is fully confirmed by the functional equation, which exhibits the mechanism of self generation of singularities.

The complete structure of singularities is still being explored in order to obtain the closest correspondence with the change of topology of the orbits of the given map.

We wish to stress that the choice of analytic area preserving maps with an elliptic fixed point is crucial for the following reason: the map depends only on the phase space coordinates r, θ and r is also the perturbation parameter. The geometry of the orbits of the real map is well understood with the help of the resonant normal forms and interpolating Hamiltonian and by complexifying the dynamics we have a map of C^2 only. Moreover these maps are the basic physical model in beam dynamics, where they describe the horizontal betratron nonlinear oscillations [11].

This investigation was inspired by Arnold's work on the circle map [12] where the analyticity problem of conjugations was raised and numerical Hénon work [13] on the quadratic map, whose improvement slowly allowed us to understand the mechanism of the series divergence and singularity generation.

Consider the map M

$$M: \begin{pmatrix} x' \\ y' \end{pmatrix} = \mathcal{R}(\omega_0) \begin{pmatrix} x + f(x, y) \\ y + g(x, y) \end{pmatrix} \tag{1}$$

where $\mathcal{R}$ is a rotation matrix, f, g are both analytic in the polydisc of radius 1 (as we can always achieve with a scaling), vanish at the origin with their first-order derivatives and satisfy the condition that makes M area-preserving. To the special class $f = 0$, $g = g(x)$ belongs the Hénon map $g = x^2$. Letting $x = \Phi(X, Y)$, $y = \Phi(X, Y)$ be a change of coordinates defined by polynomials of order N in X, Y the normal form at order N reads

$$\mathcal{M}_N: \begin{pmatrix} X' \\ Y' \end{pmatrix} = \mathbf{\Phi}^{-1} \circ M \circ \mathbf{\Phi} = \mathcal{R}(\Omega(X^2 + Y^2)) \begin{pmatrix} X + F_N(X, Y) \\ Y + G_N(X, Y) \end{pmatrix} \tag{2}$$

where $\Omega(\rho)$ is a polynomial of order $[(N-1)/2]$ in $\rho, \mathbf{\Phi} \equiv (\Phi, \Psi)$ and F_N, G_N are remainders of order $N+1$ in X, Y.

Let $F_N = \Sigma_{j+k \leqslant N} F_{j,k} X^j Y^k$ and define the norm $\|F\|_R = \Sigma_{n=2}^{N} R^n \max_{j+k=n} |F_{j,k}|$, in the polydisc $|X| \leqslant R, |Y| \leqslant R$; the following estimate then holds [15]

$$\|F_N + \mathrm{i} G_N\|_R \leqslant \left(\frac{R}{R_N}\right)^{N+1} \left(1 - \frac{R}{R_N}\right)^{-1} \qquad R_N = \frac{1}{c\gamma_0 (N+1)^{1+\eta_0}} \tag{3}$$

where we assume $\omega/2\pi$ to be diophantine $|\mathrm{e}^{ik\omega_0} - 1|^{-1} \leqslant \gamma_0 |k|^{\eta_0}$. Letting $R < R_N/2$ the norm of the remainder is bounded by 2^{-N} namely by $\exp(-[c'\gamma_0 R_N]^{-1/(1+\eta_0)})$.

Introducing polar coordinates according to $X = r\cos\theta$, $Y = r\sin\theta$ the map (2) reads

$$\mathcal{M}_N: \begin{cases} \theta' = \theta + \Omega(r^2) + a(r,\theta) \\ r' = r + b(r,\theta). \end{cases} \tag{4}$$

The functions a, b are analytic in the domain $|\mathrm{Im}\,\theta| \leqslant \Delta$, $|r| \leqslant R_N\,\mathrm{e}^{-\Delta}/4$ and are bounded by 2^{-N}; the map (4) preserves the measure $r + \mu(r,\theta)$ where μ is analytic in $|\mathrm{Im}\,\theta| \leqslant \Delta$, $|r| \leqslant R_N\,\mathrm{e}^{-\Delta}/4$ and bounded by 2^{-N}.

Before quoting the domains where the map (4) is analytically conjugated with the integrable map $\mathcal{N}$

$$\mathcal{N}: \begin{cases} \theta' = \theta + \Omega(r^2) \\ r' = r \end{cases} \tag{5}$$

we recall that the asymptotic analysis of the series shows that when the frequency Ω passes through a resonance $2\pi p/q$, then a pole appears in the conjugation function Φ. Correspondingly the perturbation series is affected by the divisor $\mathrm{e}^{\mathrm{i}q\omega} - 1$ starting at order $N \geqslant q - 1$ and behaves as a geometric series whose ratio is determined by the pole [10]. Consequently the analytic continuation by Padé approximants of the Fourier components turned out to be effective. The lowest Fourier components affected by the pole are $k = 1 \pm q$. The best way to understand the pole generation mechanism is to use the Siegel–Moser functional equations for the Fourier components. Introducing complex coordinates $z = r\,\mathrm{e}^{-\mathrm{i}\theta}$, $w = r\,\mathrm{e}^{\mathrm{i}\theta}$ and correspondingly $(\bar{f}, \bar{g}) = (f + \mathrm{i}g, f - \mathrm{i}g)$ and $(\bar{\Phi}, \bar{\Psi}) = (\Phi + \mathrm{i}\Psi, \Phi - \mathrm{i}\Psi)$ the Fourier components $\{\bar{\Phi}\}_k$ are defined by

$$\{\bar{\Phi}(\lambda z, \lambda^{-1} w)\}_k = \lambda^k \{\bar{\Phi}(z, w)\}_k \qquad \forall \lambda \in \mathrm{C}. \tag{6}$$

Observing that the normal form has only the $k = \pm 1$ Fourier components, we have the following equations

$$\begin{aligned} &\mathrm{e}^{\mathrm{i}(\Omega - \omega)} z - z = \{f(\Phi, \Psi)\}_1 \\ &\{\Phi\}_k = \frac{\{f(\Phi, \Psi)\}_k}{\mathrm{e}^{\mathrm{i}k\Omega - \omega} - 1} \qquad k \neq 1 \end{aligned} \tag{7}$$

where the choice $\{\Phi\}_1 = z$, $\{\Psi\}_{-1} = w$ has been made and the variables f, Φ, Ψ are meant to be overlined. By releasing this condition one could for instance impose that $\Omega = \omega + \Omega_2 r^2$. Equation (7) can be solved recursively starting with $\Phi = z$, $\Psi = w$, $\Omega = \omega$; one can prove that, at order n of the iteration, the functions Φ and Ω agree with the perturbation expansion up to order n. Choosing Ω to be a real polynomial of order n the sequence diverges. Letting Ω itself to be memormorphic but real analytic (real poles and real residues are allowed by the perturbative condition) a converging sequence of meromorphic functions could be obtained.

In order to show how the poles are generated we consider the second approximation where $\Omega = \omega + \Omega_2 r^2$. Observing that $\{\Phi\}_k = c_k r^{|k|}\,\mathrm{e}^{\mathrm{i}k\theta}[1 + \mathrm{O}(r)]$ for $|k| > 1$ and choosing $k = 1 \pm m$ we see that amplitude $\{\Phi\}_k$ has a pole for $(k-1)\omega + k\Omega_2 r^2 \mp 2\pi l = 0$ and more explicitly at

$$r^2 = r_m^2 \equiv \frac{m}{m \pm 1} \frac{2\pi\varepsilon_m}{-\Omega_2} \qquad \varepsilon_m \equiv \frac{\omega}{2\pi} - \frac{l}{m} \tag{8}$$

where l is the closest integer to $m\omega/2\pi$.

The poles in r can be real or imaginary depending on the sign of $-\Omega_2\varepsilon_m$. The pole for $k=1-m$ has the greatest residue γ_m with $\gamma_m=|r_m|^{m-2}/[(m-1)|\Omega_2|]$. If the pole is real, this is precisely the square of the width of the chain of islands corresponding to that resonance. Indeed using the resonant normal form with respect to l/m we find an interpolating Hamiltonian [16]

$$H=2\pi\varepsilon_m\frac{r^2}{2}+\frac{\Omega_2}{4}r^4+\ldots+Cr^m\cos(m\theta+\alpha)+\ldots \tag{9}$$

The location of hyperbolic and elliptic fixed points of h is at $r^2=2\pi\varepsilon_m/(-\Omega_2)$ which agrees with r_m up to the factor $(m-1)/m\sim 1$ and the distance of the separatrices, namely the width of the *islands* is $\Delta_m=4[C/(2\pi|\varepsilon_m|)]^{1/2}r_m^{m/2}\sim r_m^{m/2-1}$ so that $|\gamma_m|\sim\Delta_m^2$. The weight of poles is related to the hierarchy of resonances: for a generic m we have $|\varepsilon_m|\leqslant 1/(2m)$, while if m belongs to the continued fraction subsequence $\varepsilon_m\leqslant 1/m^2$.

No rigorous results are yet available on the convergence on the iterative scheme for the Siegel–Moser equations unless we are sufficiently far from the real and imaginary r-axis where uniform convergence can be obtained. The singularities generated by the sequence of meromorphic functions in the limit $n\to\infty$ are under investigation. They could be other than poles or poles accumulation points. The possibility of producing algebraic singularities is shown by the functional equation satisfied by the majorant function $W(r)$. Retaining only a finite number N of Fourier components it is possible to construct a functional equation for the majorant function [17] which, if the initial map does not contain quadratic terms (always true after one perturbation step), explicitly reads

$$W(r)=r+\frac{c^2r^2}{\bar{r}^2-W(r)} \tag{10}$$

where c is a constant and $\bar{r}^2$ a lower bound to the closest pole; using the diophantine estimate we have $\min|r_m|^2\geqslant 2\pi|\Omega_2|^{-1}\gamma_0^{-1}N^{-1}(N+1)^{-\eta_0}=\bar{r}^2$. The iterative solution of (10) has poles at finite order, while $w(r)$ itself has algebraic singularities. By refining the procedure the singularity of the majorant function appears to be located at $\bar{r}$ rather than $\bar{r}^2$ and the estimates (3) of the remainders can be improved by replacing R_N with its square root.

Using the Newton's method according to the scheme proposed by Moser we can prove the following

Theorem 1. The map M_N given by (4), analytic with its invariant measure μ in $|\mathrm{Im}\,\theta|<\Delta/2$ and $|r|<R_N\,\mathrm{e}^{-\Delta}/4$, with a, b, μ bounded by 2^{-N}, is analytically conjugated with the integrable map $\mathcal{M}$ given by (5) within the strip $|\mathrm{Im}\,\theta|\leqslant\Delta/4$ and the disc $|r|\leqslant R_N\,\mathrm{e}^{-\Delta}/8$ excluding two angular sectors of aperture χ bisected by the real and imaginary axis provided that $C2^{-N}|\tan\chi|^{-3}(\gamma_0/\Delta^{1+\eta_0})^2<1$.

As a function of the radius R_N the bound on $\tan\chi$ reads

$$|\tan\chi|\geqslant E_1(R_N)\equiv C_1\exp\left[-\left(\frac{1}{c_1\gamma_0R_N}\right)^{1/(1+\eta_0)}\right]$$

where C, C_1, c_1 are numerical constants. Taking the envelope of the angular sectors for $N\geqslant 2$ we obtain as analyticity domain the complement of the disc $|r|\leqslant R_2$ with respect to neighbourhoods of the real and imaginary axis delimited by curves having an exponential tangency order at the origin with the real and imaginary axis (see figure 1).

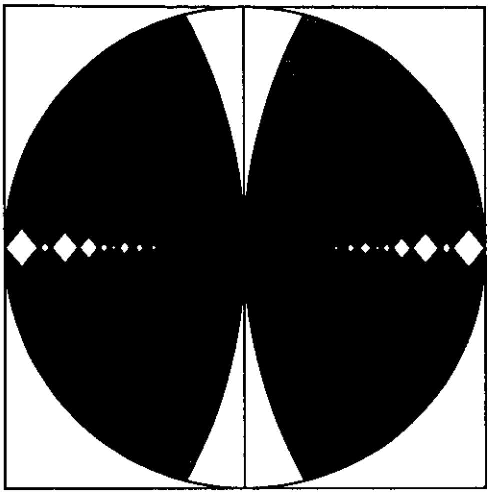

Figure 1. The analyticity domain (shaded) of the normalizing transformation in the complex radial plane. The symmetry with respect to the imaginary axis is due to the fact that the nonlinear frequency Ω (see equation (5)) is a function of r^2.

In the proof the invariance with respect to the measure $\mu(r)$ and the presence of a fixed point at the origin play an essential role to replace the intersection condition [5].

Starting from this initial analyticity domain we can perform analytic continuation to reach the real axis at points where the frequency is diophantine. We exclude neighbourhoods of the points of the real axis where the frequency is resonant.

Theorem 2. The map M_N can be analytically conjugated with the map $\mathcal{M}$ in $|\mathrm{Im}\ \theta| \leqslant \Delta/4$ and in the images by $\rho = r^2$ of the discs in the ρ plane defined by: $|\rho - \rho_c| \leqslant |\alpha|/2$ with $\rho_c = \tau + \mathrm{i}\alpha$ where $\Omega(\tau) = \omega$ provided that $C' 2^{-N}(\Gamma/\Delta^{1+\eta})^7 \leqslant 1$ where $\Gamma^{-1} = \gamma^{-1} + \nu|\alpha|$ with $\nu = \min\{1, |\Omega_2|/2\pi\}$ and γ, η are the diophantine constants of the frequency $\omega = \Omega(\tau)$.

We remark that we fix the point τ on the real axis corresponding to the choosen frequency ω, which could be non-diophantine or resonant, indeed in this case $\gamma^{-1} = 0$ and all our estimates still hold. The consequence of this theorem is that the allowed discs are given by

$$\gamma^{-1} + \nu|\alpha| \geqslant E_2(R_N) = C_2 \exp\left(-\left[\frac{1}{c_2\gamma_0 R_N}\right]^{1/(1+\eta_0)}\right).$$

At any diophantine point the conjugacy is possible provided that $\gamma^{-1} \geqslant E_2(R_N)$. In this case the analyticity domain, envelope at discs, is a cone of aperture $\pi/3$ with vertex at $\rho = \tau$ and axis parallel to the imaginary axis.

At any point τ_* such that $\Omega(\tau_*) = 2\pi p/q$ is resonant, the conjugacy is possible for $\nu|\alpha| > E_2$. Consider then the frequencies with a continued fraction expansion $\omega/(2\pi) = (a_1, \ldots, a_l, M \ldots, M \ldots)$ where $p/q = (a_1, \ldots, a_l)$ with $a_l, M \in \boldsymbol{Z}$ and observe that for M large $\gamma = M$ and $|\omega - \omega_*| = \gamma^{-1}/q^2$; as a consequence $q^2|\tau - \tau_*| + |\alpha| \geqslant \nu^{-1}E_2$ bounds the analyticity region. Combining this with the cone estimate at the end points $\tau = \tau_* + \nu^{-1}E_2$ the non-analyticity region is a rombus with the centre at $\rho = \tau_*$ and the ratio of vertical and horizontal diagonals is $\sqrt{3}$. This ratio can be optimized just as other exponents in the proof. In figure 1 a sketch of the non-analyticity domains is given.

The measure of the non-analyticity domain is proportional to $E_2(R_N)$ and the same result holds for the non-analyticity points of the real axis. This estimate exponentially small with $1/R_N$ agrees with a theorem by Neishtadt [18]. We notice that if the improved estimates on the remainders are used then E_1, E_2 are replaced with E_1^{2m}, E_2^{2m} when $\Omega=\omega+\Omega_{2m}\rho^m+\ldots$ namely if the first anysochronous term is ρ^m. Then the isochronous limit $m\to\infty$ can also be considered: the number of angular sectors in the r plane is $2m$ and covers it densely but simultaneously the measure of the non-analyticity domain vanish as $m\to\infty$. This picture is consistent with Rüssmann theorem [20] which ensures the analyticity in the full disc and similar to the one obtained in the Siegel problem [21] when the diophantine frequency is approximated with rational numbers.

To conclude this analysis we say that first integrals of motion exist for the complexified dynamics. Considering the boundary of the analyticity domain we have a Cantor set on the real axis; this allows a good understanding both of the asymptotic properties of the perturbation series and of the rise of singularities due to change of topology of the orbits. Moreover the C^∞ extensions of first integrals, which are to a large extent arbitrary, are replaced by the boundary value of the analytic first integrals which intersect the real axis in disconnected sets of large measure. The polynomial normal forms provide the required interpolation in the gaps with an error which can be made exponentially small according to the Nekhoroshev type of estimates.

A similar analysis can be carried out in the neighbourhood of an hyperbolic or complex fixed point. The normal form consists of a rotation of a complex angle; an analyticity disc around the origin exists in this case and the singularities lie on the manifold where the imaginary part of the rotation vanishes, whose distance from the origin is finite. Higher dimensional extensions are possible but the greatest difficulty in this case is the geometric interpretation of the results.

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